1 Exercises

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which
$$f(x) + y = f(y) + x$$
holds for all $x, y \in \mathbb{R}$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which
$$f(x + y) = f(x) + 2y$$
holds for all $x, y \in \mathbb{R}$.

3. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ for which
$$f(x + y) + 1 = f(x) + f(y)$$
holds for all $x, y \in \mathbb{R}$.

4. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ for which
$$y^2 f(x) = f\left(\frac{x}{y}\right)$$
holds for all $x, y \in \mathbb{R}^+$.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which
$$f(x) + y + f(xy) + 1 = f(f(x) + f(y) + f(xy + 1))$$
holds for all $x, y \in \mathbb{R}$.

6. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ for which
$$f(x) + f(y) + 2xy = f(x + y)$$
holds for all $x, y \in \mathbb{Z}$.

7. Find all injections $f : \mathbb{R} \to \mathbb{R}$ for which
$$f(x + f(y)) = f(x + y) + 1$$
holds for all $x, y \in \mathbb{R}$.

8. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(f(x) + f(y^2)) = x + f(y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is both an injection and a surjection.

9. Find all functions $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ for which
$$f(x) + 2(x - 1)f\left(\frac{x}{x - 1}\right) = 7x - 2$$
holds for all $x \in \mathbb{R} \setminus \{1\}$.

10. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function for which

$$f\left(x^2 + y\right) = f\left(x^3 + 2y\right) + f\left(x^4\right)$$

holds for all $x, y \in \mathbb{R}$. Prove that $f(x) = 0$ for every $x \in \mathbb{R}$.

11. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function for which

$$f(f(y) + xf(x)) = y + f(x)^2$$

holds for all $x, y \in \mathbb{R}$. Prove that $f(x) \in \{\pm x\}$ for every $x \in \mathbb{R}$.

2 Harder problems with some guidance

12. Recall the example shown at the start: find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$f(f(x) + y) = f(x^2 - y) + 4yf(x)$$

holds for every $x, y \in \mathbb{R}$. Recall that we proved that $f(x) = 0$ or $f(x) = x^2$ holds pointwise. Our goal is to show that this holds globally.

Suppose for contradiction that there exists $a, b \in \mathbb{R} \setminus \{0\}$ with $f(a) = a^2$ and $f(b) = 0$. By making suitable substitutions in the functional equation, derive a contradiction with the fact that $a \neq 0$ and $b \neq 0$.

13. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function which satisfies

$$f(f(y) + xf(x)) = y + f(x)^2$$

for all $x, y \in \mathbb{R}$. By using the last problem in the previous subsection and following the strategy used in the previous problem, prove that we have either $f(x) = x$ for alal $x \in \mathbb{R}$, or $f(x) = -x$ for all $x \in \mathbb{R}$

14. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ for which

$$f(f(n)) < f(n + 1)$$

holds for all $n \in \mathbb{Z}^+$.

(a) Can you prove that $f(1) = 1$?

(b) Assume that you have proved that $f(1) = 1, f(2) = 2, \ldots, f(k) = k$ for some $k \in \mathbb{Z}^+$. Can you prove that $f(k + 1) = k + 1$?

15. (Much harder problem). Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the condition

$$\frac{f(x)^2 + f(y)^2}{f(u^2) + f(v^2)} = \frac{x^2 + y^2}{u^2 + v^2}$$

for every $x, y, u, v \in \mathbb{R}^+$ with $xy = uv$.

Possible way to obtain a solution:

(a) Prove that $f(1) = 1$.

(b) By making a suitable substitution, prove that for every $x \in \mathbb{R}$ one has $f(x) = x$ or $f(x) = \frac{1}{x}$ (Hint: seek for substitution for y, u, v so that $xy = uv$, each y, u and v depends on x or is a constant, and so that the terms $f(y)$, $f(u^2)$ and $f(v^2)$ are express in terms of $f(x)$, and which gives a quadratic equation in $f(x)$).

(c) Avoid the point-wise trap: suppose for sake of contradiction that there exists $a, b \in \mathbb{R}^+ \setminus \{1\}$ so that $f(a) = a$ and $f(b) = \frac{1}{b}$. By making a suitable substitution, derive a contradiction.

(d) Conclude that there are two solutions: $f(x) = x$ for all $x \in \mathbb{R}$, or $f(x) = \frac{1}{x}$ for all $x \in \mathbb{R}$

16. (Much harder problem) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies the condition

$$f(xy) = f(x + y) + f(f(x)f(y))$$

for every $x, y \in \mathbb{R}$. Possible path to a solution:

(a) Verify that $f(x) = 0$, $f(x) = x - 1$ and $f(x) = 1 - x$ are all solutions.

(b) Find a substitution that leads to $f\left(f(x)f\left(\frac{x}{x-1}\right)\right) = 0$. Conclude that there exists $c \in \mathbb{R}$ for which $f(c) = 0$.

(c) If $f(c') = 0$ for some $c' \neq 1$, prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

(d) From now on, suppose that $f(x) = 0 \iff x = 1$. Justify why this can be assumed by the previous observations.

(e) Prove that $f(0) \in \{\pm 1\}$, and first consider the case when $f(0) = -1$.

(f) Prove that $f(x + 1) = f(x) + 1$ for all $x \in \mathbb{R}$

(g) Prove that f is an injection (this by far the hardest step. Suppose that $f(a) = f(b)$, and try to find a good substitution for x and y which allows you to use the previous bullet point).

(h) Once you have proved injectivity, substitute $y = -x$ and $y = 1 - x$ into the original equation, and use previous observations to conclude that $f(x) = x - 1$.

(i) Handle with the case when $f(0) = 1$ similarly (this leads to $f(x) = 1 - x$).